Estimating the spiking rate in systems of interacting neurons

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Interacting neurons

- N neurons $X_t^1, \ldots, X_t^N$: $X_t^i \geq 0$ membrane potential of neuron i at time t.
- Each neuron 'spikes' at rate $f_i(X_t^i)$.
- $f_i \in C^1$, strictly positive.

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- $f_i \in C^1$, strictly positive.
- If i spikes at time t :

$\Rightarrow$ neuron i is reset to a resting potential 0, i.e.

$$X_{t-}^i \to X_t^i = 0$$

$\Rightarrow$ for all $j \neq i$: j receives an additional amount of potential $W_{i \to j}$, i.e.,

$$X_{t-}^j \to X_{t-}^j + W_{i \to j}.$$

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Remark

Process is a PDMP with generator

$$Lg(x) = \sum_{i=1}^{N} f_i(x^i)[g(x + \Delta_i(x)) - g(x)] - \lambda \sum_{i=1}^{N} \frac{\partial g}{\partial x^i}(x^i - m),$$

where

$$x+\Delta_i(x) = \left(x^1 + W_{i\to 1}, \dots, x^{i-1} + W_{i\to i-1}, 0, x^{i+1} + W_{i\to i+1}, \dots\right)^T.$$

Where does this model come from ?

• Can be seen as a very easy variant of Leaky Integrate and Fire Models where spiking occurs randomly with a rate depending on the potential.

• There is some relation with interacting Hawkes processes having memory of variable length...

AIM

• Want to estimate $f_i(a)$, for $a \in \mathbb{R}_+$ fixed, based on observation of $(X^i_t)_{t\in[0,T]}$, in a non-parametric way.

• Kernel type estimator :

$$\hat{f}_{t,h}(a) := \frac{\sharp \text{ of spikes of } X^i_t, \text{ issued from positions in } B_h(a)}{\int_0^t 1_{B_h(a)}(X^i_s)ds}.$$

• Non-adaptive frame : Fix a regularity class for unknown f_i and suppose that $f_i \in H_\beta$: Hölder-class of regularity β **known** (plus some extra conditions on the growth of f_i...)

• Choose $h = h_t = t^{-\frac{1}{2\beta+1}}$ and a kernel *satisfying the usual conditions*.

Theorem (with Nathalie Krell and Pierre Hodara, SISP, 2017)

For slightly modified dynamics (compact state space !)

1)

$$\limsup_{t\to\infty} \sup_{f_i\in H(\beta)} t^{\frac{2\beta}{2\beta+1}} E_x^{f_i}[|\hat{f}_{t,h_t}(a) - f_i(a)|^2 | A_{t(h_t)} > r^*] < \infty.$$

2) If $h_t = o(t^{-\frac{1}{2\beta+1}})$,

$$\sqrt{th_t}(\hat{f}_{t,h_t}(a) - f_i(a)) \to \mathcal{N}(0, \Sigma(a)),$$

$\Sigma(a)$ *known, depends on* $f_i(a)$ *and on the invariant density in position* a.

3) Moreover, the speed of convergence $t^{\frac{2\beta}{2\beta+1}}$ *is optimal, that is,*

$$\liminf_{t\to\infty} \inf_{\hat{f}_t} \sup_{f_i\in H(\beta)} t^{\frac{2\beta}{2\beta+1}} E_x^{f_i}[|\hat{f}_{t,h_t}(a) - f_i(a)|^2] > 0.$$

Some words about the technical difficulties

No "Many-to-one"-formula ! !

Want to estimate the spiking rate of a single particle - belonging to a system of size $N-$ and subject to interactions with the rest of the system !

More precisely, need a control of

$$E\left[\int_{]s,t]}\int \bar{Q}_h(y-a)\mu^i(du,dy)|\mathcal{F}_s\right],$$

$Q_h = \frac{1}{h}Q(\frac{\cdot}{h})$, μ^i jump measure of particle i and - with π^i invariant measure of particle i,

$$\bar{Q}_h(y-a) = Q_h(y-a) - \pi^i(Q_h(\cdot - a)).$$

- So : **Need of Control of centered Additive Functionals of a single particle**

- $\|P_t(x,\cdot)-\pi\|_{TV}\leq V(x)e^{-\lambda t}$:

$V(x)$ Lyapunov function, depends on the total configuration $\Rightarrow$ **compact state space.**

- Second problem : We need : Regularity of invariant density !

Recurrence of the process

Theorem

X recurrent in the sense of Harris, with invariant probability measure π.

Follows basically from **partial regeneration** induced by spikes. (Explain on the paper board !)

Write π = unique invariant probability measure of the total system.

- $\pi^1(g) := \int_{\mathbb{R}^N} \pi(dx) g(x^1)$ invariant probability of first neuron.
- Question : $\pi(dx) = \pi(x)dx$? π smooth ? ? ? ?
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Problem : There is **not** a lot of noise in the system. Only the "exponential densities" of the jump times.

Second Problem : Jump kernel $K(x, dy) = \sum_{i=1}^N \frac{f(x^i)}{\bar{f}(x)} \delta_{x+\Delta^i(x)}(dy)$, $\bar{f} = \sum_{j=1}^N f(x^j)$, is partly degenerate ! Indeed : $[x + \Delta_i(x)]^i = 0$!!!!

IPP based on jump noise

Proposition (Resolvant formula)

Let $\gamma_t(x)$ the joint flow of the N particles starting from $x \in \mathbb{R}^N$ at time 0 (solution to ODE), $e(t,x) = e^{-\int_0^t \bar{f}(\gamma_s(x))ds}$ survival rate, $\bar{f}(x) = \sum f_i(x^i)$. Then

$$\pi(g) = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \pi(dx) f(x^i) \int_0^\infty e(t, x + \Delta_i(x)) g(\gamma_t(x + \Delta^i(x)) dt.$$

(Follows from considering the "just-before-jump" chain and its transitions)

Application

This implies for the first particle :

$$E_\pi(h'(X_t^1)) = \sum_{i=1}^N \int_{\mathbb{R}^N} \pi(dx) f(x^i) \int_0^\infty e(t, x + \Delta_i(x)) h'(\gamma_t^1(x + \Delta^i(x)))dt.$$

But $(y = [x + \Delta^i(x)]^1)$:

$$\int_0^\infty e(t,y)h'(\gamma_t^1(y))dt = \int_0^\infty \frac{e(t,y)}{b(\gamma_t^1(y))}[h \circ \gamma_t^1]'(y)dt$$

$$= \left[e(t,y)\frac{h(\gamma_t^1(y))}{b(\gamma_t^1(y))} \right]_{t=0}^{t=\infty} - \int_0^\infty \frac{d}{dt}\left(\frac{e(t,y)}{b(\gamma_t^1(y))}\right) h(\gamma_t^1(y))dt.$$

1. PROBLEM : border term at $t = 0$ gives $\frac{h(y)}{b(y)}$ where $y = [x + \Delta^i(x)]^1$ position of first neuron after a spike of i. If $i = 1$ this gives the total contribution

$$\int \pi(dx) f(x^1) \frac{h(0)}{b(0)} : \text{ Dirac measure in 0!}$$

$\Longrightarrow$ have to stay away from 0 ! ! !

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2. PROBLEM : we divide by $b(\gamma_t^1(y))$.
$\Longrightarrow$ have to stay away from $\{y : b(y) = 0\} = \{m\}$.

Theorem (EL, to appear in SPA)

Let $f_i \in H(\beta)$, $\beta = k + \alpha, \alpha \in [0, 1[$, for all $1 \le i \le N$. Suppose moreover that $\|f_i\|_{\infty,k} \le F$. EL, Then

$$\pi^1 \in C^k(\Omega_k)$$

and

$$\sup_{v \neq v', v, v' \in \Omega_k} \frac{|(\pi^1)^{(k)}(v) - (\pi^1)^{(k)}(v')|}{|v - v'|^\alpha} \le C,$$

where C does not depend on f_i, but only on the bounds of the function class f belongs to.

Here, Ω_k denotes the subset of all positions "sufficiently far away" from 0 and from m, **even after k IPP's !**

Outlook : Lebesgue density in dimension N

First comments :

- Since we can only use the jump noise, we need at least N jumps.
- The flow transports (preserves) density nicely.
- Jump of particle i destroys density in direction of e_i.

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First comments :

- Since we can only use the jump noise, we need at least N jumps.

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- Jump of particle i destroys density in direction of e_i. But :

Immediately after, density is created by the jump noise.

NUMMELIN SPLITTING

For the "just-before-jump"-chain $Z_k = X_{T_k -}$, with associated transition kernel Q :

Theorem

$$Q^N(x, dy) \geq 1_C(x)\beta\nu(y)dy,$$

where $\nu \in C_c^\infty(\mathbb{R}^N)$.

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• Idea of proof : A specific **order of successive jumps** creates density : e.g. 1 spikes first, followed by 2 followed by 3 etc... this could be compared to the weak Hörmander condition. Successive jumps of $1, 2, 3, \ldots$ induce a diagonal structure of what would be the "Malliavin covariance matrix" here.

• The idea of using favorable sequences of jump events has already been used by Duarte and Ost (2015) to show Harris recurrence of the process.

• Nummelin splitting implies : there exists an extended stopping time (the regeneration time) R such that

$$X_{T_R-} = Z_R \sim \nu(x)dx.$$

• Can we preserve this density ? ? ?

Preservation of density

We start from

$$X_{T_{R-}} = Z_R \sim \nu(x)dx.$$

Suppose i jumps at time $T_R \Rightarrow$ replace $x \mapsto x + \Delta_i(x)$: does not depend on x^i any more. But :

$$(t, x) \mapsto G(t, x) = \gamma_t(x + \Delta_i(x)) \text{ explores the whole space :}$$

$$J_G(t, x) = \det \sqrt{\frac{\partial G}{\partial t \partial x}\left(\frac{\partial G}{\partial t \partial x}\right)^T} > 0.$$

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In this case, the **co-area formula** implies that we have a measurable Lebesgue density for Z_{R+1} and thus for Z_n for all $n \geq R$. In particular, $\pi(dx) = \pi(x)dx$ with some **measurable** π.

In order to obtain more regularity, we have to work more (no IPP, but transformations of variables - based on the flow for the non-spiking particles, and based on the jump noise for the spiking one) :

Theorem (EL, to appear in SPA)

If $f_i \geq f_0 > \lambda$, the invariant density π is at least $k-$times differentiable, for any $k : 2k < Nf_0/\lambda - N$.

So we need a balance between the explosion rate λ of the inverse flow and the minimal jump rate.

This is of course a very strong condition - but the transitions are also very degenerate...

Some literature

• Duarte, A., Ost, G. A model for neural activity in the absence of external stimuli. To appear in Markov Proc. Related Fields 2016, available on http ://arxiv.org/abs/1410.6086.

• Poly, G. Absolute continuity of Markov chains ergodic measures by Dirichlet forms methods. To appear in Ann. IHP, 2013.

• You can find the statistical work with Nathalie and Pierre on arXiv : https ://arxiv.org/abs/1604.07300. it will soon appear in SISP.

• And the work on the regularity of the invariant density will appear in SPA, see also https ://arxiv.org/abs/1601.07123.

Thank you for your attention.